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2007 J. Phys. A: Math. Theor. 40 4997

(http://iopscience.iop.org/1751-8121/40/19/005)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 4997–5012 doi:10.1088/1751-8113/40/19/005

The response of subdiffusive bistable fractional Fokker–Planck systems to rectangular signals

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Received 2 September 2006, in final form 10 January 2007
Published 24 April 2007
Online at stacks.iop.org/JPhysA/40/4997

Abstract

We study the response of one-dimensional subdiffusive fractional Fokker–Planck systems with a general confining potential, when it is perturbed from its stationary state by a time-dependent non-sinusoidal driving force. Three types of rectangular driving signals have been investigated: a rectangular pulse, a periodic telegraph signal and a generalized telegraph signal with a fractional duty cycle. We derive analytic expressions for the linear response and the input energy in one period of the driving signal. In particular, for signals with a long period, we obtain several asymptotic results concerning the wave form of the response and the stochastic energetics. Numerical results for representative symmetric, as well as asymmetric, subdiffusive bistable systems are presented and discussed.

PACS numbers: 05.10.Gg, 45.10.Hj

1. Introduction

Dynamical processes of complex systems often exhibit intriguing features including, for example, anomalous diffusive behaviour, non-exponential relaxation, non-Gaussian probability density functions and long-range spatial or temporal correlations obeying peculiar power laws; hence, they have received much attention in recent years. In order to describe the anomalous dynamical processes which occur in a variety of physical and biological systems, fractional calculus has been found to be a useful tool, and various fractional dynamical equations have been proposed [1–3]. In this work, we shall focus on the dynamics described by the non-Markovian subdiffusive fractional Fokker–Planck equation (SFFPE) [2].

A diversity of phenomena displays subdiffusive dynamical behaviour. For example, it is observed in the charge carrier transport in amorphous semiconductors, the spread of contaminants in underground water, nuclear magnetic resonance diffusometry in percolative systems and porous systems, motions of polymeric systems, as well as protein conformational dynamics [2]. The SFFPE, with its characteristic slowly decaying kernel which generates strong memory effects, provides a theoretical framework to describe subdiffusion. Being an extension of the classical Fokker–Planck equation, the SFFPE can be derived by several approaches [4–7], and it can also be regarded as the overdamped case of the fractional Klein–Kramers equation [2, 3, 7, 8]. Recently, this equation has been applied to a number of problems which reveal the significance of subdiffusive dynamics, including dynamics of annealed systems [9], linear response in complex systems [10], diffusion on comb-like structure [11], subdiffusive motion of bistable systems [12], anomalous transport in tilted periodic potentials [13], subdiffusive reaction–diffusion processes [14–20] and fluorescence lifetime for small single molecules [21, 22].

In this work we shall study the behaviour of a one-dimensional subdiffusive fractional Fokker–Planck system, with a confining potential and initially in its stationary state, when it is perturbed by a time-dependent driving force. We consider three types of non-sinusoidal driving forces: a rectangular pulse, a periodic telegraph signal and a generalized telegraph signal with a fractional duty cycle. In [23–27], the effects of rectangular driving signals on a diffusive noisy bistable system have been investigated by analogue simulations and by solving the Langevin equation numerically. In our work, the focus is on the consequences of the subdiffusive dynamics, and we shall limit our discussion to linear response theory only. Besides the response to the driving signals, we also examine the stochastic energetics of the system [28]. For the case of a Brownian bistable system perturbed by a sinusoidal time-varying force, the work done by the external agent on the system displays resonance-type dependence on the noise intensity as well as the frequency of the driving force [29–31]. It is of interest to perform a comparative study of the subdiffusive counterpart in the presence of perturbing rectangular pulses. For a general confining potential, we shall derive analytic expressions for the linear response of the system and the externally injected energy in one period of the driving signal. When the driving force has a long period, these formulae allow us to deduce several asymptotic results concerning the secular behaviour of the system. Numerical results for representative double-well confining potentials will be presented to confirm the validity of the theoretical analyses and to illustrate the effects of the noise intensity and the relative duration of the pulses.

This paper is organized as follows. In section 2, the linear response theory for a subdiffusive fractional Fokker–Planck system with a confining potential is presented. In section 3, we consider the linear response to a rectangular signal of a finite duration. In section 4, we study the linear response to a periodic telegraph signal. In particular, for a signal with a long period, we show that the theoretical result is in accord with the adiabatic approximation, and we obtain an asymptotic expression of the secular linear response. Numerical results for representative subdiffusive bistable systems are presented and discussed. In section 5, the linear response to a telegraph signal with a fractional duty cycle is analysed. We show that the dependence of the secular stochastic energy on the duty cycle satisfies a complementarity relation. Section 6 contains our concluding remarks.

2. Basic equations and the linear response theory

We begin with the one-dimensional SFFPE of the probability density function (PDF), $P(x,t)$, for the position x at time t, of a particle moving in a static confining potential $U(x)$ with $-\infty < x < \infty$ [4]:

$$\frac{\partial}{\partial t} P(x,t) = {}_0\hat{D}_t^{1-\gamma}\big[\hat{L}_{\mathrm{FP}}^0(x) P(x,t)\big], \tag{1}$$

where the fractional Riemann–Liouville operator is defined by

$$ {}_0\hat{D}_t^{1-\gamma}[f(x,t)] = \frac{1}{\Gamma(\gamma)}\frac{\partial}{\partial t}\left[\int_0^t (t-t')^{\gamma-1} f(x,t')\,\mathrm{d}t'\right], \tag{2} $$

with $0 < \gamma < 1$ and Γ denotes the gamma function. The Fokker–Planck operator $\hat{L}_{\rm FP}^0$ contains a positive diffusion constant D and the gradient of the potential $U'(x) = [\mathrm{d}U(x)/\mathrm{d}x]$ [32]:

$$ \hat{L}_{\rm FP}^0(x) = D\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}U'(x). \tag{3} $$

For a confining potential, $U = \infty$ and $P = 0$ at $x = \pm\infty$. We assume that all quantities in equation (1) are expressed in appropriate dimensionless units [3]. The stationary solution of equation (1) is given by the distribution [4, 32]

$$ P_0(x) = N\exp[-U(x)/D], \tag{4} $$

where N is a normalization constant. The eigenfunctions $\{\varphi_n(x)\}$, with eigenvalues $\{-\lambda_n\}$, of the Fokker–Planck operator constitute a convenient set of basis functions for the SFFP system

$$ \hat{L}_{\rm FP}^0(x)\varphi_n(x) = -\lambda_n\varphi_n(x), \qquad n = 0, 1, 2, \ldots, \tag{5} $$

with $\lambda_0 = 0$ and $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$. Through the transformation $\psi_n(x) = \varphi_n(x)\exp[U(x)/2D]$, we arrive at a pseudo-Schrö dinger equation, with a D-dependent potential, for ψ_n [32]:

$$ \hat{H}\psi_n(x) = (\lambda_n/2D)\psi_n(x), \tag{6} $$

where $\hat{H} = \frac{1}{2}\hat{p}_x^2 + V_D(x)$, $\hat{p}_x = -\mathrm{i}(\mathrm{d}/\mathrm{d}x)$ and $V_D(x) = [U'(x)]^2/(8D^2) - U''(x)/(4D)$. With ψ_n taken to be real and normalized, we have the orthonormality relation

$$ \int_{-\infty}^{\infty}\psi_m(x)\psi_n(x)\,\mathrm{d}x = \int_{-\infty}^{\infty}\varphi_m(x)\varphi_n(x)\exp[U(x)/D]\,\mathrm{d}x = \delta_{mn}. \tag{7} $$

In particular, we note that the 'ground-state wavefunction' is given by $\psi_0(x) = \sqrt{P_0(x)}$, and $\int_{-\infty}^{\infty}\varphi_n(x)\,\mathrm{d}x = \delta_{n0}/\sqrt{N}$.

We introduce several matrix elements and sums which are essential to our analysis of the dynamical response problem. We define $X_{mn} = \int_{-\infty}^{\infty}\psi_m(x)x\psi_n(x)\,\mathrm{d}x \equiv \langle\psi_m|x|\psi_n\rangle$, $M_{n0} = \langle\psi_n|[-U'(x)/D]|\psi_0\rangle$ and $C_j = -\sum_{n=1}^{\infty}X_{0n}M_{n0}\lambda_n^{-j}$ with $j = 1, 2$. Because of the relation $-U'(x)\psi_0(x) = 2D\psi_0'(x)$ and the commutation relation $[x, \hat{H}] = \mathrm{i}\hat{p}_x$, we can transform M_{n0} as

$$ M_{n0} = 2\langle\psi_n|[x,\hat{H}]|\psi_0\rangle = -(\lambda_n/D)X_{n0}. \tag{8} $$

Thus, $M_{00} = 0$. Furthermore, using the closure relation $\sum_{n=0}^{\infty}|\psi_n\rangle\langle\psi_n| = 1$ and integration by parts, we obtain the following sum rules:

$$ \sum_{n=1}^{\infty}X_{0n}M_{n0} = \int_{-\infty}^{\infty}x\frac{\mathrm{d}}{\mathrm{d}x}\left[\psi_0^2(x)\right]\mathrm{d}x = -1, \tag{9} $$

$$ \sum_{n=1}^{\infty}X_{0n}M_{n0}\lambda_n = -\langle\psi_0|[U'(x)]^2|\psi_0\rangle/D, \tag{10} $$

and

$$ C_1 = [\langle\psi_0|x^2|\psi_0\rangle - \langle\psi_0|x|\psi_0\rangle^2]/D. \tag{11} $$

Therefore, C_1 is a positive quantity: it is determined by the variance of x in the ground state ψ_0 of $\hat{H}$. We note that, by equation (8), C_2 is also found to be positive:

$$C_2 = \sum_{n=1}^{\infty} X_{0n}^2 \lambda_n^{-1} D^{-1}. \tag{12}$$

For the SFFP system, the transition probability density can be written as an eigenfunction expansion [4, 14, 32]

$$P(x, t|x', t') = \sum_{n=0}^{\infty} \frac{\psi_0(x)}{\psi_0(x')} \psi_n(x)\psi_n(x') E_\gamma[-\lambda_n (t - t')^\gamma], \tag{13}$$

where $t > t'$ and $E_\gamma(z) = \sum_{j=0}^{\infty} z^j / \Gamma(1 + \gamma j)$ is the Mittag–Leffler function. The above formula can be used to calculate the stationary auto-correlation function $C_{xx}(t)$ [12] which, for $t > 0$, is defined as

$$\begin{aligned} C_{xx}(t) &= \int_{-\infty}^{\infty} x P(x, t|x', 0) x' P_0(x') \, \mathrm{d}x \, \mathrm{d}x' - \left[\int_{-\infty}^{\infty} x P_0(x) \, \mathrm{d}x \right]^2 \\ &= \sum_{n=1}^{\infty} X_{0n}^2 E_\gamma(-\lambda_n t^\gamma). \end{aligned} \tag{14}$$

The frequency transform of C_{xx} yields the spectral density $S_{xx}(\omega)$ [33]:

$$S_{xx}(\omega) \equiv 2 \int_0^{\infty} C_{xx}(t) \cos \omega t \, \mathrm{d}t = \left[\frac{2 \sin\left(\frac{1}{2}\gamma\pi\right)}{\omega^{1-\gamma}} \right] \sum_{n=1}^{\infty} \frac{X_{0n}^2 \lambda_n}{Y_n^2(\omega, \gamma)}, \tag{15}$$

where a characteristic function for the spectral property of the SFFP system has been introduced:

$$Y_n(\omega, \gamma) = \left[\omega^{2\gamma} + 2\omega^\gamma \lambda_n \cos\left(\tfrac{1}{2}\gamma\pi\right) + \lambda_n^2\right]^{\frac{1}{2}}. \tag{16}$$

We now turn to the problem of dynamical linear response. Suppose that the particle is initially in the stationary state described by the PDF $P_0(x)$ shown in equation (4). For $t > 0$, let a spatially homogeneous but time-dependent force be applied to the particle so that its potential energy becomes $U(x, t) = U(x) + H(t)x$, where $-H(t)$ represents a weak time-dependent driving force. We write the PDF at $t > 0$ as $P(x, t) = P_0(x) + P_1(x, t)$ and calculate P_1 to the first order of $H(t)$. P_1 obeys the equation of motion

$$\frac{\partial}{\partial t} P_1(x, t) = {}_0\hat{D}_t^{1-\gamma} \left[\hat{L}_{\mathrm{FP}}^0(x) P_1(x, t) + P_0'(x) H(t)\right], \tag{17}$$

with the initial condition $P_1(x, 0) = 0$. It is helpful to consider the Laplace transform of equation (17) with respect to t:

$$s^\gamma \tilde{P}_1(x, s) = \hat{L}_{\mathrm{FP}}^0(x) \tilde{P}_1(x, s) + P_0'(x) \tilde{H}(s). \tag{18}$$

To solve equation (18), we write $\tilde{P}_1$ as $\tilde{P}_1(x, s) = \sum_{n=1}^{\infty} \tilde{c}_n(s) \varphi_n(x)$; the exclusion of $\varphi_0(x)$ ensures that $\int_{-\infty}^{\infty} P_1(x, t) \, \mathrm{d}x = 0$. The orthonormality relation shown in equation (7) allows us to determine the expansion coefficients. Finally, we obtain P_1 in the form of an eigenfunction-expansion

$$P_1(x, t) = \sum_{n=1}^{\infty} \sqrt{N} M_{n0} a_n(t; \gamma) \varphi_n(x). \tag{19}$$

The time-dependent coefficient a_n, given by the inverse Laplace transform

$$a_n(t; \gamma) = L^{-1}[(s^\gamma + \lambda_n)^{-1} \tilde{H}(s)] \equiv L^{-1}[\tilde{F}_n(s; \gamma) \tilde{H}(s)], \tag{20}$$

is expressible as a convolution integral

$$a_n(t;\gamma) = \int_0^t F_n(t-t';\gamma)H(t')\,\mathrm{d}t', \tag{21}$$

where the function $F_n(t;\gamma)$ can be related to the derivative of a Mittag–Leffler function

$$F_n(t;\gamma) = L^{-1}[(s^\gamma + \lambda_n)^{-1}] = -\lambda_n^{-1}[\mathrm{d}E_\gamma(-\lambda_n t^\gamma)/\mathrm{d}t]. \tag{22}$$

The average value of x at $t > 0$ is given by $\langle x(t)\rangle = X_{00} + \Delta x_1(t)$ with the linear response determined by

$$\Delta x_1(t) = \int_{-\infty}^{\infty} x P_1(x,t)\,\mathrm{d}x = \sum_{n=1}^{\infty} X_{0n} M_{n0} a_n(t;\gamma). \tag{23}$$

Equations (19)–(23) are the basic results of our linear response theory for a confined SFFP system. We next proceed to investigate the linear response to three types of rectangular signals: (1) a rectangular pulse, (2) a periodic telegraph signal and (3) a generalized telegraph signal with a fractional duty cycle. For these cases, the integral in equation (21) can be calculated, and we obtain analytic expressions of $\Delta x_1(t)$.

3. Response to a rectangular pulse

Consider the perturbation due to a rectangular pulse switched on at time T_0 and off at time T_1: $H(t) = -A\Theta(T_1 - t)\Theta(t - T_0)$, where A is a constant and Θ is Heaviside's step function.

By equations (21)–(23), the linear response for $T_0 < t < T_1$ is found to be

$$\Delta x_1(t) = -A\sum_{n=1}^{\infty} X_{0n} M_{n0}\lambda_n^{-1}[1 - e_n(\gamma, t - T_0)], \tag{24}$$

where we have introduced the notation $e_n(\gamma,t) \equiv E_\gamma(-\lambda_n t^\gamma)$. It is of interest to examine the response immediately after the pulse is turned on (i.e., $t \to T_0^+$). Using the series representation of E_γ in conjunction with the sum rules shown in equations (9) and (10), we find that the early-time response has the following asymptotic behaviour:

$$\begin{aligned}\Delta x_1(t) &= A\left[\frac{(t-T_0)^\gamma}{\Gamma(1+\gamma)} - \frac{\langle\psi_0|[U'(x)]^2]|\psi_0\rangle}{D\Gamma(1+2\gamma)}(t-T_0)^{2\gamma} + O(t-T_0)^{3\gamma}\right]\\ &\equiv \Delta x_a(t - T_0).\end{aligned} \tag{25}$$

It is noteworthy that the first term in Δx_a in fact does not depend on the potential $U(x)$ and is solely characterized by the subdiffusiveness parameter γ. We obtain a divergent derivative $[\mathrm{d}\Delta x_1(t)/\mathrm{d}t] \simeq A/[\Gamma(\gamma)(t-T_0)^{1-\gamma}]$ when $t \to T_0^+$.

For $t > T_1$, the linear response is given by

$$\Delta x_1(t) = -A\sum_{n=1}^{\infty} X_{0n} M_{n0}\lambda_n^{-1}[e_n(\gamma, t - T_1) - e_n(\gamma, t - T_0)], \tag{26}$$

which depends on the time differences $(t - T_0)$ and $(t - T_1)$. Such temporal dependence has been found in previous works on subdiffusive systems without a confining potential [9, 10]. It is shown in these works that the linear response of systems modelled by continuous-time random walks is different and exhibits aging. Immediately after the cessation of the perturbation (i.e., $t \to T_1^+$), we find that

$$\Delta x_1(t) \simeq -A\sum_{n=1}^{\infty} X_{0n} M_{n0}\lambda_n^{-1}[1 - e_n(\gamma, T_1 - T_0)] - \Delta x_a(t - T_1), \tag{27}$$

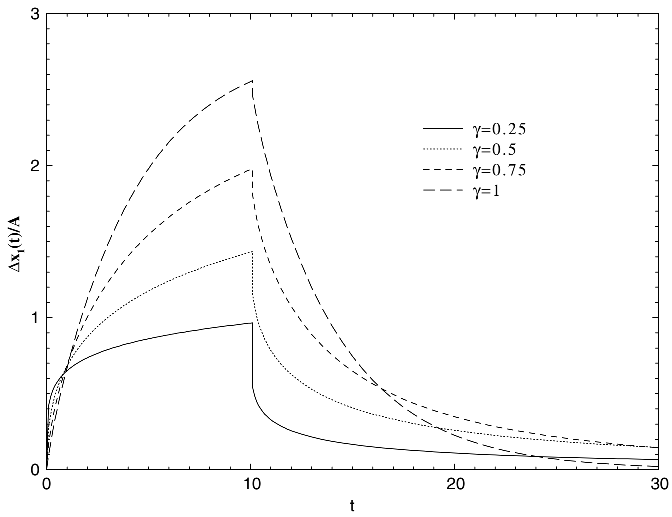

Figure 1. The time dependence of the response $[\Delta x_1(t)/A]$ to a rectangular pulse with $T_0 = 0$ and $T_1 = 10$ for the potential U_4 at $D = 0.3$ with different values of γ.

where Δx_a has been defined in equation (25). Comparing the responses given by equations (25) and (27), we see that the latter has an increment given by the first term in equation (27), which is positive if $A > 0$. The second term in equation (27) is related to the early-time response close to T_0 by a sign reversal and a temporal translation from T_0 to T_1. When $t \to T_1^+$, we again have a divergent derivative $[\mathrm{d}\Delta x_1(t)/\mathrm{d}t] \simeq -A/[\Gamma(\gamma)(t-T_1)^{1-\gamma}]$.

In the long-time limit, the linear response is proportional to the total impulse $A \times (T_1 - T_0)$ delivered by the pulse and decays algebraically with increasing time

$$\Delta x_1(t) \simeq [\gamma C_2/\Gamma(1-\gamma)][A \times (T_1 - T_0)/t^{1+\gamma}], \qquad t \to \infty. \tag{28}$$

This result is obtained by applying the asymptotic formula $E_\gamma(-z) \simeq [\Gamma(1-\gamma)z]^{-1}$ for large z to equation (26).

Thus we encounter the characteristic constant C_2 in the long-time tail of the linear response. To illustrate the aforementioned features, we show in figure 1 the response of a system with the symmetric double-well potential $U_4(x) = (-0.5x^2 + 0.25x^4)$ and $D = 0.3$, for different degrees of subdiffusiveness. Our eigenfunctions and eigenvalues are obtained by the state-dependent diagonalization method [34]. The series shown in equations (24) and (26) are found to converge rapidly. The case with $\gamma = 1$ is the conventional Fokker–Planck system whose response decays exponentially with increasing time for large t.

4. Response to a telegraph signal

We now study the effect of a telegraph signal, with a period $2T$ and an amplitude A (> 0), defined by

$$H(t) = -A \sum_{k=0}^{\infty} (-1)^k \Theta(t - kT)\Theta(kT + T - t). \tag{29}$$

We recall that the telegraph signal has been investigated in the problem of coherent stochastic resonance of diffusive systems with two absorbing boundaries [35]; its effect on the mean survival time does not display a resonance-like dependence on the frequency, in marked contrast with a sinusoidal periodic driving force [35, 36].

At time $t = (KT + \epsilon T)$, where K is a positive integer and $0 \leqslant \epsilon < 1$, equations (21)–(23) yield

$$\Delta x_1(t = KT + \epsilon T) = A \times (-1)^K \sum_{n=1}^{\infty} X_{0n} M_{n0} \lambda_n^{-1} \left\{ S_n(\gamma, \epsilon T) + (-1)^K e_n(\gamma, (KT + \epsilon T)) + \sum_{j=1}^{K-1} (-1)^j 2 e_n(\gamma, (jT + \epsilon T)) \right\}, \quad (30)$$

where $S_n(\gamma, \epsilon T) = [2e_n(\gamma, \epsilon T) - 1]$. We are interested in the secular response with $K \gg 1$ for which equation (30) can be approximated by

$$\Delta x_{1s}(t = KT + \epsilon T) = A \times (-1)^K \sum_{n=1}^{\infty} X_{0n} M_{n0} \lambda_n^{-1} \left\{ S_n(\gamma, \epsilon T) + \sum_{j=1}^{\infty} (-1)^j 2 e_n(\gamma, (jT + \epsilon T)) \right\}. \quad (31)$$

We have performed numerical calculations based on equation (30) for the symmetric double-well potential $U_4(x)$ and the asymmetric double-well potential $U_3(x, a_3) = (-0.5x^2 + a_3 x^3 + 0.25x^4)$. When $a_3 = 0.4$, U_3 is highly asymmetrical: it has a deeper well around $x = -1.766$, a shallow well around $x = 0.566$, and the ratio of the two potential-well depths is 21:1.

In figure 2, we depict the results of the response for different parameters. We observe that for $1 \geqslant D \geqslant 0.1$ and $T < 10$, the wave form of the evolution of the secular response is rather different from that of a telegraph signal and the dissimilarity increases as the subdiffusive character of the system decreases. However, if the period of the driving signal is increased at a fixed value of D, then for sufficiently large T the wave form of the response gradually resembles the shape of a telegraph signal. This phenomenon has also been observed in recent works which investigate the nonlinear response of the diffusive system with the potential U_4 by solving the Langevin equation numerically [26, 27].

To understand this long-period effect, we introduce a characteristic time scale of the SFFP system: $T^*(\gamma) = \lambda_1^{-1/\gamma}$, which provides a useful gauge for the period of the driving signal. T^* is sensitive to the symmetry of the confining potential and the values of D and γ. For example, for U_4 and $D = 0.05$, we have $T^*(1) = 360$ and $T^*(0.5) = 1.30 \times 10^5$; however, if D is increased to 0.1, T^* is markedly reduced: $T^*(1) = 29.8$ and $T^*(0.5) = 889$. In contrast, for U_3 with $a_3 = 0.4$ and $D = 0.1$, we find $T^*(1) = 8.75$ and $T^*(0.5) = 76.5$.

For a telegraph signal with a very long period, the last sum in equation (31) becomes negligible since it is of the order of $T^{-\gamma}$, and the quantity $S_n(\gamma, \epsilon T)$ thus plays an important role. Let us assume that the value of $\epsilon (0 < \epsilon < 1)$ satisfies the condition $g \equiv [\epsilon T / T^*(\gamma)]^\gamma \gg 1$. Then $e_n(\gamma, \epsilon T) = O(\lambda_1 / \lambda_n g) \ll 1$, $S_n(\gamma, \epsilon T) \simeq -1$, and, therefore, $\Delta x_{1s}(KT) \simeq -\Delta x_{1s}(KT + \epsilon T)$ since $S_n(\gamma, 0) = +1$. If the condition $g \gg 1$ is met even when ϵ is quite small, then the secular response will resemble a telegraph signal. For example, for the case of U_4 with $D = 0.5$, $\gamma = 1$ and $T = 100$, we find, for $\epsilon = 0.1$, that $S_1 = -0.97$ and $S_n = -1.00$ for $n \geqslant 2$. However, if γ is reduced to 0.5, then for the same $\epsilon = 0.1$ we need $T = 1.5 \times 10^4$ to yield $S_1 = -0.931$ and $S_2 = -0.987$.

In order to gain further understanding of the long-time behaviour of $\Delta x_1(t)$, we calculate the inverse Laplace transform shown in equation (20) by means of the Bromwich–Hankel path in the complex s-plane [37, 38]. The Laplace transform of the telegraph signal is

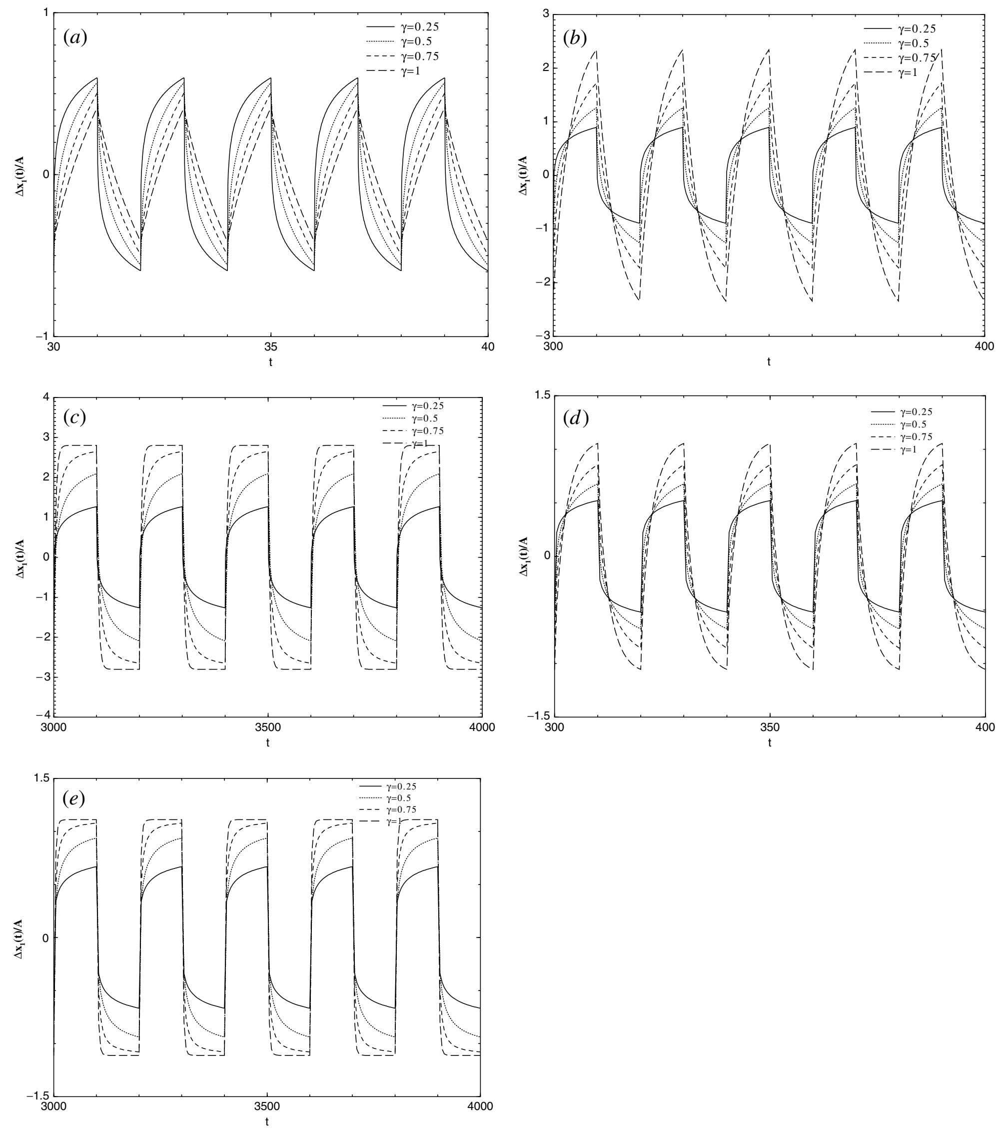

Figure 2. The response $[\Delta x_1(t)/A]$ to a telegraph signal of period $2T$ for different values of γ. Parts (*a*)–(*c*) are for U_4 at $D = 0.3$ and $T = 1, 10, 100$, respectively. Parts (*d*) and (*e*) are for U_3 with $a_3 = 0.4$ at $D = 0.5$ and $T = 10, 100$, respectively.

$\tilde{H}(s) = -(A/s)\tanh(Ts/2)$. For $0 < \gamma < 1$, the expansion coefficient $a_n(t;\gamma)$ is composed of two parts. The first part is the contribution from integrals along the two borders of the cut negative real axis of the complex s-plane:

$$[a_n(t;\gamma)]_{\text{cut}} = \int_0^\infty \tilde{H}(-s)K_n(s,\gamma)\exp(-st)\,\mathrm{d}s, \tag{32}$$

with the kernel given by

$$K_n(s,\gamma) = \frac{s^\gamma \sin\gamma\pi}{\pi\left(s^{2\gamma} + 2s^\gamma\lambda_n\cos\gamma\pi + \lambda_n^2\right)}. \tag{33}$$

The second part comes from the residues of the poles of $\tilde{F}_n(s;\gamma)\tilde{H}(s)\exp(st)$:

$$[a_n(t;\gamma)]_{\rm res} = -\frac{4A}{\pi}\sum_{j=0}^{\infty}\frac{\sin[\Omega_j t - \delta_n(\Omega_j,\gamma)]}{(2j+1)Y_n(\Omega_j,\gamma)}, \tag{34}$$

where $\Omega_j = (2j+1)\pi/T$, Y_n has been defined in equation (16) and

$$\delta_n(\Omega_j,\gamma) = \arctan\left[\frac{\Omega_j^{\gamma}\sin\frac{1}{2}\gamma\pi}{\lambda_n + \Omega_j^{\gamma}\cos\frac{1}{2}\gamma\pi}\right]. \tag{35}$$

For $0 < \gamma < 1$, the simple poles of $\tilde{F}_n$ are not situated in the main Riemann sheet and, therefore, do not contribute to the sum of residues [37].

Accordingly, $\Delta x_1(t) = [\Delta x_1(t)]_{\rm cut} + [\Delta x_1(t)]_{\rm osc}$, where the second term, associated with the coefficients given in equation (34), can be written as a Fourier series

$$[\Delta x_1(t)]_{\rm osc} = \sum_{j=0}^{\infty}[\chi(\Omega_j)\exp({\rm i}\Omega_j t) + {\rm c.c}], \tag{36}$$

with the complex Fourier coefficients given by

$$\chi(\Omega_j) = \frac{2{\rm i}A}{\pi(2j+1)}\sum_{n=1}^{\infty}\frac{X_{on}M_{no}}{Y_n(\Omega_j,\gamma)}\exp[-{\rm i}\delta_n(\Omega_j,\gamma)]. \tag{37}$$

When $t\to\infty$, $[\Delta x_1(t)]_{\rm cut} \simeq (AC_2T/2\pi)(\sin\gamma\pi)\Gamma(1+\gamma)t^{-(1+\gamma)}$ and eventually becomes negligible. For the case of $\gamma = 1$, equation (36) remains valid, but $[\Delta x_1(t)]_{\rm cut}$ is absent and, instead, we have another term $\sum_{n=1}^{\infty}X_{0n}M_{n0}\tilde{H}(-\lambda_n)\exp(-\lambda_n t)$ which decays exponentially for large t.

Now, consider a telegraph signal with a long period. We can estimate the asymptotic behaviour of equation (36) in the limit $[\pi T^*(\gamma)/T]^{\gamma} \to 0$ by using the small-Ω_j approximation for equation (37): $\chi(\Omega_j) \simeq [2{\rm i}A/\pi(2j+1)]\sum_{n=1}^{\infty}X_{on}M_{no}\lambda_n^{-1}$, which yields $[\Delta x_1(t)]_{\rm osc} \simeq (4AC_1/\pi)\sum_{j=0}^{\infty}(\sin\Omega_j t)/(2j+1)$. Noting that the telgraph signal has the Fourier series representation $H(t) = -(4A/\pi)\sum_{j=0}^{\infty}(\sin\Omega_j t)/(2j+1)$, we arrive at the asymptotic relation $[\Delta x_1(t)]_{\rm osc} \simeq -C_1H(t)$ when $[T/\pi T^*(\gamma)]^{\gamma} \to \infty$. Thus, the wave pattern of the secular response and the driving force $-H(t)$ are nearly the same. Recalling the numerical dependence of T^* on γ and D, we conclude that for the normal diffusive system with $\gamma = 1$ the long-period condition is easier to fulfil. It is interesting to note that the constant C_1, which is closely related to the uncertainty of x in the ground state ψ_0, serves as the asymptotic proportionality constant. For U_4 with $D = 0.3$, we find that $C_1 = 2.7$, and the theoretical asymptotic relation is in good agreement with the numerical findings shown in figure 2(*c*).

The above result of the secular response is reminiscent of the adiabatic approximation. Therefore, we perform an analogous analysis of the PDF.

When a similar long-period approximation is applied to equation (34), we obtain $[a_n(t;\gamma)]_{\rm res} \simeq \lambda_n^{-1}H(t)$ which, by equation (19), leads to $P_1(x,t) \simeq D^{-1}H(t)(X_{00}-x)P_0(x)$ for large t. It can be verified that this P_1 indeed yields the asymptotic secular linear response mentioned above. The corresponding PDF is given by $P(x,t) = P_0(x) + P_1(x,t) \simeq N\exp\{[H(t)X_{00} - U(x) - H(t)x]/D\}$ correct to $O(H)$. We see that this expression can be interpreted as a simple adiabatic approximation to the secular PDF for a driving force which varies very slowly with time, since the initial unperturbed stationary PDF is given by $P_0(x) = N\exp[-U(x)/D]$: replacing $U(x)$ by $[U(x) + H(t)x]$ in $P_0(x)$ generates the above approximate $P(x,t)$ up to a normalization constant. We emphasize that this adiabatic approximation is valid when $[\pi T^*(\gamma)/T]^{\gamma} \ll 1$.

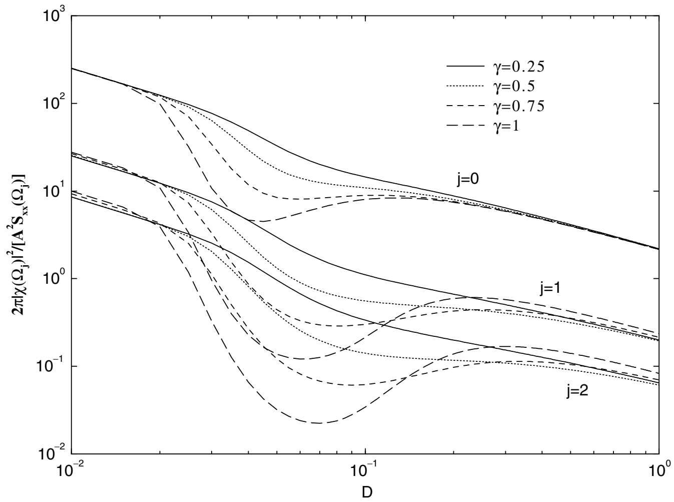

Figure 3. The D-dependence of the ratio $2\pi|\chi(\Omega_j)|^2/[A^2 S_{xx}(\Omega_j)]$ of U_4 with $T = 100$ for $j = 0, 1, 2$.

In figure 3, we depict the D-dependence of the ratio $2\pi|\chi(\Omega_j)|^2/[A^2 S_{xx}(\Omega_j)]$ for the potential U_4 with $T = 100$. We only show the results for $j = 0$, 1 and 2 since they are the predominant frequency components in equation (36). The above ratio can be regarded as a measure of the output signal-to-noise ratio for each frequency component of the secular response. We observe that its variation with D is apparently non-monotonic when $\gamma \geqslant 0.75$.

We next examine the energetics of the system. In the time interval (t_1, t_2), the energy supplied to the particle and the heat bath by the external agent that causes the temporal fluctuations of the potential is given by [28–31]

$$W(t_1, t_2, \gamma) = \int_{t_1}^{t_2} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x \left[\frac{\partial U(x,t)}{\partial t}\right] P(x,t) = \int_{t_1}^{t_2} \left[\frac{\mathrm{d}H(t)}{\mathrm{d}t}\right] \langle x(t)\rangle \,\mathrm{d}t. \tag{38}$$

For a telegraph signal with a period $2T$, it is convenient to take $kT < t_1 < (kT + T)$, where k is a positive integer, and $t_2 = (t_1 + 2T)$, so as to study the input energy in one period. For linear response, $\langle x(t)\rangle = X_{00} + \Delta x_1(t)$. After integrations by parts and making use of the characteristics of a telegraph signal, we find

$$W(t_1, t_1 + 2T, \gamma) = (-1)^{k+1}(2A)[\Delta x_1(kT + 2T) - \Delta x_1(kT + T)].$$

Applying the formula of Δx_1 shown in equation (30), we obtain

$$W(t_1, t_1 + 2T, \gamma) = -4A^2 \sum_{n=1}^{\infty} X_{on} M_{no} \lambda_n^{-1} \left[1 + 2\sum_{j=1}^{k} (-1)^j e_n(\gamma, jT) + \frac{(-1)^k}{2} e_n(\gamma, kT + 2T) - (-1)^k \frac{3}{2} e_n(\gamma, kT + T)\right], \tag{39}$$

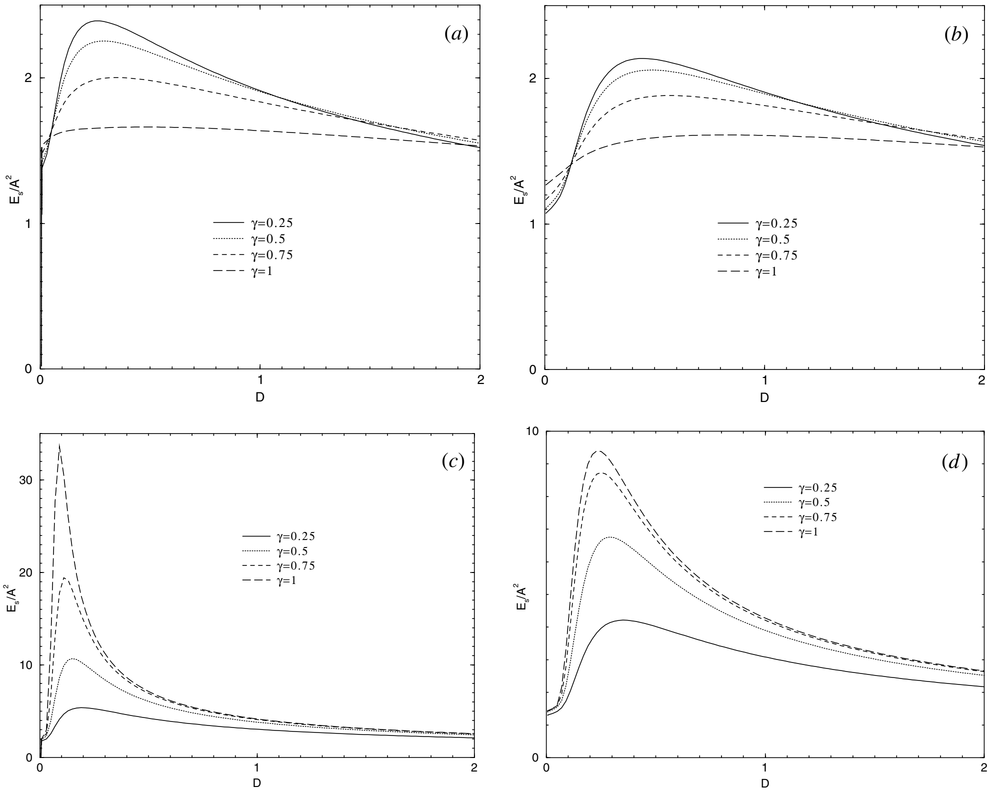

Figure 4. The D-dependence of (E_s/A^2) for a telegraph signal of period $2T$. Parts (*a*) and (*c*) are for U_4 at $T = 1$ and $T = 100$, respectively. Parts (*b*) and (*d*) are for U_3 with $a_3 = 0.2$ at $T = 1$ and $T = 100$, respectively.

which, in the large-k limit, yields the secular value of the energy input in one period of the driving signal

$$E_s(\gamma, T) = -4A^2 \sum_{n=1}^{\infty} X_{on} M_{no} \lambda_n^{-1} \left[1 + 2 \sum_{j=1}^{\infty} (-1)^j e_n(\gamma, jT) \right]. \tag{40}$$

For the normal diffusive system with $\gamma = 1$, the sum over j can be calculated. We obtain

$$\frac{E_s(\gamma = 1, T)}{A^2} = 4C_1 - \frac{8}{D} \sum_{n=1}^{\infty} \frac{X_{0n}^2}{1 + \exp(\lambda_n T)}. \tag{41}$$

On the other hand, for a SFFP system perturbed by a telegraph signal with a long period $[T/T^*(\gamma)]^\gamma \gg 1$, equation (40) yields the asymptotic expansion

$$\frac{E_s(\gamma, T)}{A^2} \simeq 4C_1 - \frac{8f(\gamma)}{\Gamma(1-\gamma)} \frac{C_2}{T^\gamma}, \tag{42}$$

where $f(\gamma) = \sum_{j=1}^{\infty} (-1)^{j+1} j^{-\gamma}$ is a positive numerical constant. Although equations (41) and (42) indicate the same limiting value $E_s \to 4A^2C_1$ as $T \to \infty$, the subdiffusive system will approach this limit at a much slower rate as T increases.

In figure 4, we show the D-dependence of $E_s(\gamma, T)$ for the potentials U_4 and U_3 with $a_3 = 0.2$. For given values of γ and T, E_s shows a maximum at an optimal value of D. For large T, the peak of E_s becomes more distinct when γ is increased, whereas the introduction of asymmetry in the confining potential significantly reduces the sharpness and the height of the peak.

5. Response to telegraph signal with fractional duty cycle

Consider a driving signal of the following form:

$$H(t) = -A\sum_{k=0}^{\infty}(-1)^k\Theta(t-kT)\Theta[kT+rT-t], \qquad 0<r<1. \tag{43}$$

The parameter r is referred to as the duty cycle [23–25, 27]. There is a time lapse of $(1-r)T$ between the cessation of one pulse and the advent of the following pulse. The telegraph signal discussed in section 4 corresponds to $r=1$.

For $KT<t<(KT+rT)$, where $K=0,1,2,3,\ldots$, a combination of equations (21), (22) and (43) yields

$$\lambda_n a_n(t;\gamma)/A = \sum_{k=0}^{K-1}(-1)^k[e_n(\gamma,t-kT)-e_n(\gamma,t-kT-rT)] + (-1)^K[e_n(\gamma,t-KT)-1]. \tag{44}$$

If $(K-1+r)T<t<KT$, where $K=1,2,3,\ldots$, we have

$$\lambda_n a_n(t;\gamma)/A = \sum_{k=0}^{K-1}(-1)^k[e_n(\gamma,t-kT)-e_n(\gamma,t-kT-rT)]. \tag{45}$$

The response $\Delta x_1(t)$ can be calculated according to equation (23). In figure 5, we show the results for U_4 in response to a driving signal with $T=100$ and $r=0.5$, for different values of D. When $[(1-r)T/T^*(\gamma)]^\gamma \gg 1$, the particle has a sufficiently long time to relax in the pulse-free time interval, and there will be negligible residual response remaining at the beginning of the ensuing pulse. This is seen to be the case for $D=0.3$ and $\gamma=1$.

For $r\neq 1$, $H(t)$ has the Fourier series representation $H(t) = -\frac{4A}{\pi}\sum_{j=0}^{\infty}\frac{1}{2j+1}\sin\left(\frac{1}{2}r\Omega_j T\right)\cos\left(\Omega_j t-\frac{1}{2}r\Omega_j T\right)$. By the contour integration method mentioned in the previous section, we get $[\Delta x_1(t)]_{\mathrm{osc}} = \sum_{j=0}^{\infty}2|\chi(\Omega_j)|\sin\left(\frac{1}{2}r\Omega_j T\right)\cos\left(\Omega_j t-\frac{1}{2}r\Omega_j T-\phi_j\right)$ where $\phi_j = \arctan[-\mathrm{Re}\,\chi(\Omega_j)]/[-\mathrm{Im}\,\chi(\Omega_j)]$. We again find the asymptotic relation $[\Delta x_1(t)]_{\mathrm{osc}} \to -C_1 H(t)$ when $[T/\pi T^*(\gamma)]^\gamma \to \infty$.

Taking $kT<t_1<(kT+rT)$, we calculate the input energy $W(t_1,t_1+2T,\gamma;r)$ in a manner similar to that for the telegraph signal. The result is

$$\begin{aligned} W(t_1,t_1+2T,\gamma;r)/2A^2 = -\sum_{n=1}^{\infty}X_{on}M_{no}\lambda_n^{-1}\Bigg\{ & 1+(-1)^{k+1}e_n(\gamma,kT+T) \\ & +2\sum_{j=1}^{k}(-1)^j e_n(\gamma,jT)+\sum_{j=0}^{k}(-1)^j[e_n(\gamma,jT+T-rT)-e_n(\gamma,jT+rT)] \\ & +\frac{(-1)^k}{2}[e_n(\gamma,kT+2T)-e_n(\gamma,kT+2T-rT) \\ & +e_n(\gamma,kT+T+rT)-e_n(\gamma,kT+T)]\Bigg\}. \end{aligned} \tag{46}$$

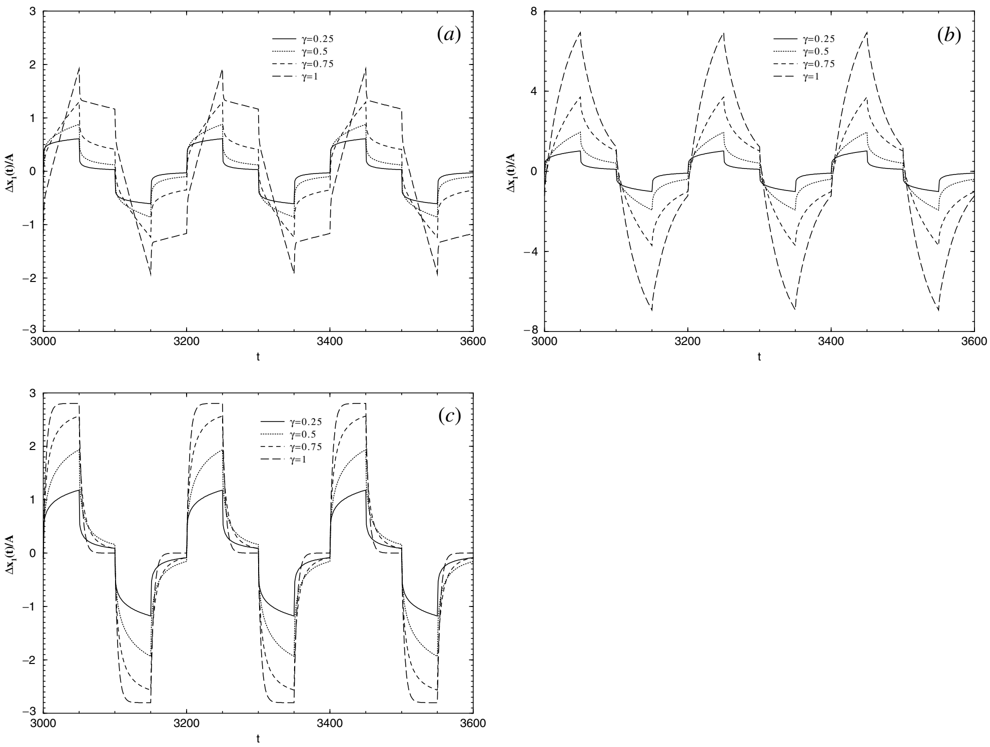

Figure 5. The response $[\Delta x_1(t)/A]$ to a telegraph signal of half period $T = 100$ and duty cycle $r = 0.5$ for the potential U_4 with different values of γ: (*a*) $D = 0.05$, (*b*) $D = 0.1$, (*c*) $D = 0.3$.

In the large-k limit, we obtain the secular value

$$\frac{E_s(\gamma, T; r)}{2A^2} = -\sum_{n=1}^{\infty} X_{on} M_{no} \lambda_n^{-1} \left\{ 1 + 2\sum_{j=1}^{\infty} (-1)^j e_n(\gamma, jT) + \sum_{j=0}^{\infty} (-1)^j [e_n(\gamma, jT + T - rT) - e_n(\gamma, jT + rT)] \right\}. \tag{47}$$

Upon setting $r = 1$ in equation (47), we find that $E_s(\gamma, T; r = 1)$ reproduces the result of the telegraph signal shown in equation (40). Furthermore, equation (47) leads to the complementarity relation

$$E_s(\gamma, T; r) + E_s(\gamma, T; 1 - r) = E_s(\gamma, T; r = 1), \tag{48}$$

for $0 < r < 1$, as one may expect. In particular, we have $2E_s(\gamma, T; r = 0.5) = E_s(\gamma, T; r = 1)$.

For $\gamma = 1$, the r-derivative of equation (47) is found to be

$$E_r(\gamma = 1, T; r) \equiv \mathrm{d}[E_s(\gamma = 1, T; r)/A^2]/\mathrm{d}r = -2T \sum_{n=1}^{\infty} X_{on} M_{no} [\mathrm{e}^{-(1-r)\lambda_n T} + \mathrm{e}^{-r\lambda_n T}]/(1 + \mathrm{e}^{-\lambda_n T}). \tag{49}$$

Several conclusions can be drawn when $\lambda_1 T \gg 1$. We note that in the range of r that satisfies $r\lambda_1 T \gg 1$ and $(1 - r)\lambda_1 T \gg 1$, the above r-derivative is exponentially small, indicating

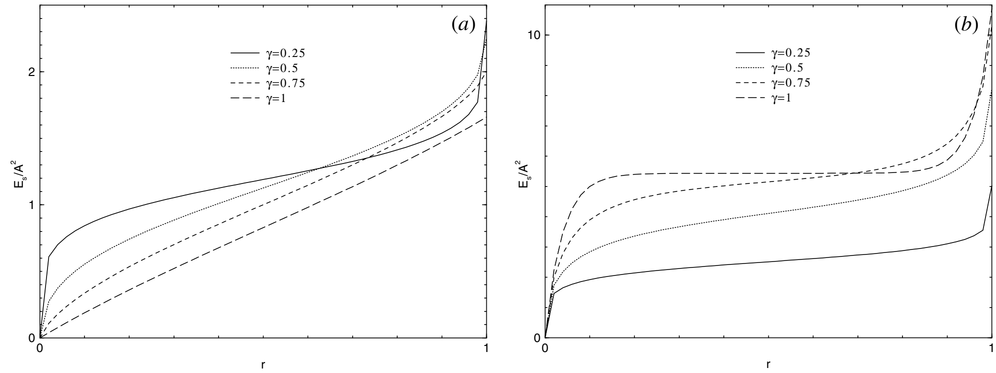

Figure 6. The r-dependence of $E_s(\gamma, T; r)/A^2$ for U_4 at $D = 0.31$ with different values of γ: (a) $T = 1$, (b) $T = 100$.

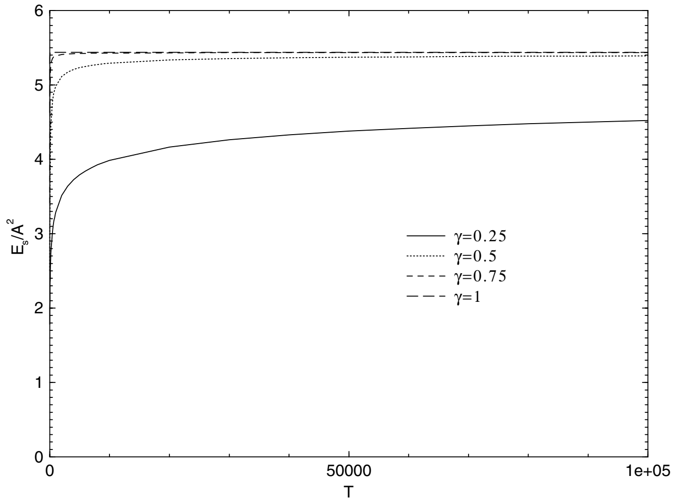

Figure 7. The T-dependence of $E_s(\gamma, T; r = 0.5)/A^2$ for U_4 at $D = 0.31$ with different values of γ.

that E_s has a weak dependence on r. Since the factors $\exp(-\lambda_n T)$ in equation (49) are now negligible, we obtain

$$E_r(\gamma = 1, T; r \to 0^+) \simeq E_r(\gamma = 1, T; r \to 1^-) \simeq 2T, \tag{50}$$

where we have used equation (9). Similar analysis for the second-order r-derivative yields the asymptotic result

$$E_{rr}(\gamma = 1, T; r \to 0^+) \simeq -E_{rr}(\gamma = 1, T; r \to 1^-) \simeq -\frac{2T^2}{D}\langle\psi_0|[U'(x)]^2|\psi_0\rangle, \tag{51}$$

where we have used equation (10). Therefore, for a driving signal with a sufficiently long period, $E_s(\gamma = 1, T; r)$ rises rapidly at $r = 0^+$ and $r = 1^-$. For intermediate values of r, we expect to see a plateau in the r-variation of E_s. As can be shown by means of equation (47), the value of E_s is about $2A^2C_1$ when $r\lambda_1 T = O(1)$, whereas E_s is close to $4A^2C_1$ at $r = 1$. These features are illustrated in figure 6(b).

For $\gamma < 1$, if $\lambda_1 r^\gamma T^\gamma \gg 1$ and $\lambda_1(1-r)^\gamma T^\gamma \gg 1$, equation (47) yields

$$\frac{E_s(\gamma, T; r)}{A^2} \simeq 2C_1 + \frac{C_2}{\Gamma(1-\gamma)T^\gamma}\left\{-4f(\gamma) + \sum_{j=0}^{\infty}(-1)^j\left[\frac{2}{(j+1-r)^\gamma} - \frac{2}{(j+r)^\gamma}\right]\right\}. \quad (52)$$

A comparison of equation (52) with equation (42) shows that in this range of r, $E_s(\gamma, T; r < 1) = 0.5E_s(\gamma, T; r = 1) + O(T^{-\gamma})$.

Figure 7 shows the T-dependence of $E_s(\gamma, T; r = 0.5)$ for U_4 with $D = 0.31$. We see that for a subdiffusive system with $\gamma = 0.25$, (E_s/A^2) deviates noticeably from the asymptotic limit $2C_1 = 5.44$ even when the period of the driving signal is as long as 2×10^6.

6. Conclusion

By the eigenfunction expansion method, we have studied the perturbing effects of periodic rectangular pulses on one-dimensional subdiffusive fractional Fokker–Planck system with a confining potential and initially in the stationary state. We have derived analytic expressions for the linear response and the externally injected energy per period and performed asymptotic analyses when the driving signal has a long period.

It is useful to compare the response to a sinusoidal driving force $H(t) = -A\sin\omega t$ with the response to a telegraph signal. For the former case [33], we obtain $[\Delta x_1(t)]_{\mathrm{osc}} = 2|\chi_s(\omega)|\sin(\omega t - \phi)$, where $\chi_s(\omega) = \frac{\mathrm{i}A}{2}\sum_{n=1}^{\infty}\frac{X_{0n}M_{n0}}{Y_n(\omega,\gamma)}\exp[-\mathrm{i}\delta_n(\omega,\gamma)]$ and the phase lag is given by $\phi = \arctan\{[-\mathrm{Re}\,\chi_s(\omega)]/[-\mathrm{Im}\,\chi_s(\omega)]\}$. The wave form of the secular linear response in this case is sinusoidal irrespective of the value of ω, although the amplitude and the phase lag have complicated dependence on D, γ and ω. As to the input energy in one period of oscillation $(2\pi/\omega)$, we obtain $E_s = -2\pi A\,\mathrm{Re}[\chi_s(\omega)]$ which exhibits a non-monotonic dependence on the frequency: $E_s \sim \omega^\gamma$ when $0 < \omega^\gamma \ll \lambda_1$, $E_s \sim \omega^{-\gamma}$ when $\omega \to \infty$, and E_s has a maximum at some intermediate optimal value of the driving frequency [33]. In contrast, since the telegraph signal is formed by a special superposition of sinusoidal waves with discrete characteristic frequencies, the associated $E_s(\gamma, T; r = 1)$ shows an entirely disparate dependence on the period of the signal: E_s increases with increase in T and $E_s = 4A^2C_1 - O(T^{-\gamma})$ for large T. Moreover, as shown in this work, it is only when $[T/\pi T^*(\gamma)]^\gamma \gg 1$ that the temporal variation of the secular linear response of a subdiffusive system resembles a telegraph signal. On the other hand, as far as the noise intensity dependence is concerned, E_s displays the resonance-like variation with D for both kinds of driving signals.

The results of our linear response theory of subdiffusive systems are complementary to those of recent works [23–27] which probe the nonlinear response of diffusive systems by simulations. The complicated nonlinear response problem of the SFFP systems deserves further investigations.

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